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Envelope solitons of nonlinear Schrödinger equation with an anti-cubic nonlinearity

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Abstract
On the basis of a recently-proposed method to find solitary solutions of generalized nonlinear Schrödinger equations (Fedele R and Schamel H 2002 *Eur. Phys. J.* B **27** 313, Fedele R 2002 *Phys. Scr.* **65** 502, Fedele R, Schamel H and Shukla P K 2002 *Phys. Scr.* T **98** 18), the existence of envelope solitonlike solutions of a nonlinear Schrödinger equation containing an anti-cubic nonlinearity ($|\Psi|^{-4}\Psi$) plus a 'regular' nonlinear part is investigated. In particular, in the case that the regular nonlinear part consists of a sum of cubic and quintic nonlinearities (i.e. $q_1|\Psi|^2\Psi + q_2|\Psi|^4\Psi$), an upper-shifted bright envelope solitonlike solution is explicitly found.

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Let us consider the following generalized Korteweg–de Vries equation (gKdVE):

$$a\frac{\partial u}{\partial s} - G[u]\frac{\partial u}{\partial x} + \frac{\nu^2}{4}\frac{\partial^3 u}{\partial x^3} = 0 \tag{1}$$

where a and ν are real constants, and $G[u]$ is a real functional of u, and the following generalized nonlinear Schrödinger equation (gNLSE):

$$\mathrm{i}\alpha\frac{\partial \Psi}{\partial s} + \frac{\alpha^2}{2}\frac{\partial^2 \Psi}{\partial x^2} - U[|\Psi|^2]\Psi = 0 \tag{2}$$

where $U[|\Psi|^2]$ is a real functional of $|\Psi|^2$ and α is a real constant.

Recently, it has been shown that a correspondence between (1) and (2) has been constructed in such a way that, provided that the following conditions are satisfied, i.e.

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$$\nu = \alpha \tag{3}$$

$$u_0 a = -u_0^2 + 2c_0 \tag{4}$$

$$G[u] = u\frac{\mathrm{d}U[u]}{\mathrm{d}u} + 2U[u] \tag{5}$$

if $u(x - u_0 s) \equiv u(\xi)$ (u_0 being a real constant) is a non-negative stationary-profile solution of (1), thus

$$\Psi(x, s) = \sqrt{u(\xi)}\exp\left\{\frac{\mathrm{i}}{\alpha}\left[\phi_0 - c_0 s + u_0\xi + A_0\int\frac{\mathrm{d}\xi}{u(\xi)}\right]\right\} \tag{6}$$

is a stationary-profile envelope solution of (2), where ϕ_0, c_0 and A_0 are real constants (not all independent) [2]. Note that $u(\xi) = |\Psi(x, s)|^2$. This result has been successfully applied to find new analytical solitary-wave envelope solutions of some modified nonlinear Schrödinger equations (MNLSE) containing high-order nonlinearities (with respect to the standard cubic nonlinearity) [2, 3]. In particular, analytical solutions in the form of bright and dark/grey envelope solitonlike solutions have been found for a MNLSE with cubic plus quintic nonlinear terms[6], i.e.

$$\mathcal{U}(|\Psi|^2) = q_1|\Psi|^2 + q_2|\Psi|^4. \tag{7}$$

These solutions have recently been used for describing the envelope soliton formation in the interaction of an intense light beam with a plasma when high-order effects are taken into account [5]. In such kinds of problems, the nonlinear coupling between light beams and non-resonant ion density perturbation causes the relativistic particle mass to increase as well as the ponderomotive effect (light pressure). These two competing effects can produce, in turn, the dependence of the refractive index on the light intensity which plays the role of an effective nonlinear potential of a suitable MNLSE governing the propagation of light beams. The nonlinear potential accounts for cubic as well as quintic nonlinearities, such as the one defined by equation (7). Under suitable conditions, these two terms become of the same order and their competition makes possible stationary structures in the form of bright and dark/grey envelope solitons [5]. Similar physical circumstances have recently been investigated in nonlinear optics where a multidimensional cubic–quintic NLSE governs the dynamics of both localized vortex solitons [6]. The (2 + 1)-dimensional cubic–quintic NLSE has been used for a stability analysis of spinning ring solitons [7]. Furthermore, for this type of NLSE, criteria for the existence and stability of soliton solutions have been formulated [8] and both modulational instability analysis and stability analysis of bright envelope solitons in the strong interaction of a light beam with plasmas have also been carried out [5]. Remarkably, algebraic solitary-wave solutions [9] and some other classes of travelling wave solutions [10] have been found for the cubic–quintic NLSE.

In this paper, we use the method developed in [1–3] to find envelope solitonlike solutions of the following MNLSE containing, besides the cubic and quintic nolinearities, an 'anti-cubic' nonlinearity (i.e. $|\Psi|^{-4}\Psi$), namely, we assume that the functional U has the form

$$U(|\Psi|^2) = Q_0|\Psi|^{-4} + q_1|\Psi|^2 + q_2|\Psi|^4 = Q_0|\Psi|^{-4} + \mathcal{U}(|\Psi|^2) \tag{8}$$

where Q_0 is a real constant. Our aim is to use mapping (6) to find envelope solitonlike solutions of the following MNLSE:

$$\mathrm{i}\alpha\frac{\partial\Psi}{\partial s} + \frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - [Q_0|\Psi|^{-4} + q_1|\Psi|^2 + q_2|\Psi|^4]\Psi = 0. \tag{9}$$

[6] It is worth mentioning that dark solitary waves have also been found in [4] by reducing the NLSE to the KdVE, but in the limit of small amplitudes only, whilst the connection between NLSE and KdVE described in [1–3] was constructed for arbitrary amplitudes and nonlinearities.

Our mathematical problem is connected with some aspect of the above correspondence between (1) and (2) developed in [1–3]. In fact, once the gKdVE (1) is given and solitonlike solutions are found, the associated gNLSE, whose solutions are given by means of (6), is obtained through conditions (3)–(5). In particular, by assuming the functional $G[u]$ to be known, we find the general solution of the differential equation (5) which allows us to get the explicit form of the functional $U[|\Psi|^2]$ to be inserted in the associated gNLSE, namely,

$$U[u] = \frac{1}{u^2}\left[K_0 + \int G[u]u\,\mathrm{d}u\right] \tag{10}$$

where K_0 is an arbitrary real constant. Under suitable conditions of regularity of $G[u]$, the criteria for the existence of bright and dark/grey stationary-profile envelope solitons of the associated gNLSE for the cases $K_0 = 0$ and $K_0 \neq 0$ have already been established [2], and explicit solitonlike solutions have been found for the case $K_0 = 0$ with cubic–quintic nonlinearities [2, 3, 5].

Note that, if we choose

$$G[u] = 3q_1u + 4q_2u^2 \tag{11}$$

according to the above correspondence, equation (10) gives us ($u = |\Psi|^2$)

$$U[|\Psi|^2] = K_0|\Psi|^{-4} + q_1|\Psi|^2 + q_2|\Psi|^4 \tag{12}$$

which justifies why, provided that K_0 is replaced by Q_0, we are going to solve equation (9). The role of K_0 is related to an interesting mathematical feature. In fact, mapping (6) allows us to construct solitonlike solutions of (9) which have the same functional form as those satisfying (2) with U given by equation (7) (i.e. $K_0 = 0$). The only restriction for (9) is that $u(\xi)$ must not vanish somewhere, which implies that we have to impose that $u(\xi)$ be a positive solitonlike solution of the following MKdVE of the type (1) with the nonlinearity given by (11):

$$a\frac{\partial u}{\partial s} - [3q_1u + 4q_2u^2]\frac{\partial u}{\partial x} + \frac{\nu^2}{4}\frac{\partial^3 u}{\partial x^3} = 0. \tag{13}$$

In fact, the vanishing of u corresponds to a divergence of the nonlinear potential term $|\Psi|^{-4}$ in (9). This circumstance excludes the standard dark solitonlike solutions.

On the other hand, it can be shown that [2]

$$-\frac{\nu^2}{2}\frac{\mathrm{d}^2u^{1/2}}{\mathrm{d}\xi^2} + \frac{Q_0}{u^{3/2}} + \frac{1}{u^{3/2}}\int G[u]u\,\mathrm{d}u = \left(c_0 + \frac{u_0^2}{2}\right)u^{1/2} - \frac{A_0^2}{2u^{3/2}}. \tag{14}$$

It is then clear from (14) that, for $Q_0 \neq 0$, a family of solitary wave solutions of (9) can be obtained by imposing the following condition:

$$A_0 = \pm\sqrt{-2Q_0} \tag{15}$$

which implies that such a kind of family of solutions exists for negative values of Q_0. In fact, condition (15) selects the suitable values of the arbitrary constant A_0 for finding solitonlike solutions, by providing the cancellation of the term $Q_0/u^{3/2}$ on the left-hand side of (14) with $A_0^2/u^{3/2}$ on the right-hand side. Consequently, equation (14) becomes the following NLSE for stationary states:

$$-\frac{\alpha^2}{2}\frac{\mathrm{d}^2u^{1/2}}{\mathrm{d}\xi^2} + \mathcal{U}[u]u^{1/2} = E_0u^{1/2} \tag{16}$$

where $E_0 = c_0 + u_0^2/2$ and $\mathcal{U}$ still have the functional form given by equation (7). Although $Q_0 \neq 0$, equation (16) admits solitary solutions which formally coincide with the ones found in [2] for $Q_0 = 0$. Thus, we can conclude that for any

$$Q_0 < 0 \tag{17}$$

under condition (15), solitary-wave solutions of (9) can be directly and explicitly constructed from (3)–(6) by using the explicit form of the positive solitonlike solutions of MKdVE (13) as given in [2], i.e.

$$\Psi_{\pm}(x,s) = \sqrt{\overline{u}\left[1+\epsilon\,\mathrm{sech}\left(\frac{\xi}{\Delta}\right)\right]}\exp\left\{\frac{\mathrm{i}}{\alpha}\left[\phi_0 - \left(E_0 - \frac{u_0^2}{2}\right)s + u_0\xi \pm \sqrt{2|Q_0|}\int\frac{\mathrm{d}\xi}{\sqrt{\overline{u}\,[1+\epsilon\,\mathrm{sech}\,(\xi/\Delta)]}}\right]\right\} \tag{18}$$

where, in principle, according to [2], ϵ should be taken in the following range:

$$-1 < \epsilon \leqslant 1 \tag{19}$$

which excludes the standard 'dark' solitary waves ($\epsilon = -1$), namely, the condition for which the modulus of Ψ vanishes at $\xi = 0$.

Actually, the direct substitution of $u = |\Psi|^2$ given by (18) into the eigenvalue equation (16) allows us to find

$$\epsilon = 1 \tag{20}$$

$$\overline{u} = -\frac{3q_1}{8q_2} \tag{21}$$

$$q_1 > 0 \qquad q_2 < 0 \tag{22}$$

$$E_0 = -\frac{15q_1^2}{64q_2} \tag{23}$$

$$\Delta = \frac{2|\alpha|}{q_1}\sqrt{\frac{2|q_2|}{3}}. \tag{24}$$

Consequently, solution (18) can be cast as

$$\Psi_{\pm}(x,s) = \sqrt{\frac{3q_1}{8|q_2|}\left[1+\mathrm{sech}\left(\frac{\xi}{\Delta}\right)\right]}\exp\left\{\frac{\mathrm{i}}{\alpha}\left[\phi_0 - \left(\frac{15q_1^2}{64|q_2|} - \frac{u_0^2}{2}\right)s + u_0\xi \pm \sqrt{2|Q_0|}\frac{16|\alpha||q_2|}{3q_1^2}\sqrt{\frac{2|q_2|}{3}}\left(\frac{\xi}{\Delta} - \tanh\left(\frac{\xi}{2\Delta}\right)\right)\right]\right\} \tag{25}$$

where u_0 is a fully arbitrary soliton velocity. According to the classification of the solitary waves used in [2], equation (25) represents an upper-shifted bright envelope solitonlike solution of equation (9), provided that the coefficients Q_0, q_1 and q_2 satisfy conditions (17) and (22), respectively. It is clear that equation (20), which does not contradict condition (19), also implies that grey solitary solutions do not exist in the solution form (18).

In conclusion, in this paper we have used a recently-developed method for solving a wide family of modified nonlinear Schrödinger equations containing high-order nonlinearities [1–3], based on knowledge of the solutions of the associated modified Korteweg–de Vries equation. An upper-shifted bright envelope solitonlike solution of a cubic–quintic NLSE containing, additionally, an 'anti-cubic' nonlinear term (see equation (9)) has been found. We have outlined the mathematical reasons that have naturally led us to consider such an equation. Furthermore, our analysis has shown that, once the form of the envelope solitonlike solution of (9) is taken according to (18), dark and grey envelope solitary solutions do not exist; in contrast, they exist in the case that the anti-cubic term is missing.

We would like to point out that although the inclusion of the anti-cubic nonlinearity in equation (9) was not suggested by the need of solving a special physical problem, very rapid developments that have taken place in the last few decades in nonlinear optics may open some new interesting perspectives, especially in the experimental studies of nonlinear materials, such as polydiaceliteles [11] which exhibit a refractive index whose dependence on the light intensity is rather complex [12]. For instance, the one-dimensional electronic delocalization due to the solid-state polymerization of diacetylenes produces dramatic enhancement of the optical nolinearities [13], and high-order nonlinear susceptibility of poly(3-butylthiophene) and poly(3-decylthiophene) has been widely studied [14].

Moreover, the results we have obtained in this paper may also deserve attention for a recently-developed investigation on the tomographic representation of solitons which makes use of the marginal distribution associated with the NLSE [15].

Finally, from a purely mathematical point of view, equation (9) may also deserve attention for specific investigations, already on the way, of its properties of symmetry and the applications of the recursion operator.

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